General idea: stochastic process $x(t)$ $\Rightarrow$ time evolution of $P(x,t)$ the proba that $x(t)=x$.

I) Langevin and Fokker-Planck

$$\dot{x}_i(t) = f_i(\vec{x}) + \eta_i(t) \quad \text{where } \eta_i \text{ GWN, centred, } \langle \eta_i(t)\eta_j(t')\rangle = 2\sigma\,\delta(t-t')\,\delta_{ij}$$

a) a useful trick

$$P(x,t) = \int dx_0\, \delta(x-x_0)\, P(x_0,t)$$

$$= \langle \delta(x - x_0(t))\rangle_{x_0(t)}$$

$$P(\vec{x},t) = \langle \delta(\vec{x} - \vec{x}_0(t))\rangle_{\vec{x}_0(t)} \qquad \delta(\vec{x}) = \prod_i \delta(x_i)$$

b) Itō calculus

$$\frac{\partial}{\partial x} f(x-y) = -\frac{\partial}{\partial y} f(x-y)$$

$$\frac{d}{dt} P(\vec{x},t) = \left\langle \frac{d}{dt}\, \delta(\vec{x} - \vec{x}_0(t))\right\rangle_{\vec{x}_0(t)}$$

$$\frac{d}{dt}\prod_i \delta(y_i - x_i(t)) = \sum_{j=1}^{N}\left\{ \underbrace{\frac{\partial}{\partial x_j}\left[\prod_i \delta(y_i - x_i(t))\right]}_{-\frac{\partial}{\partial y_j}\left[\prod_i \delta(y_i - x_i(t))\right]} \cdot \dot{x}_j(t) + \frac{1}{2}\, 2\sigma\, \frac{\partial^2}{\partial x_j^2} \prod_i \delta(y_i - x_i(t))\right\}$$

$$= -\sum_j \frac{\partial}{\partial y_j}\left\{ \underbrace{\prod_i \delta(y_i - x_i(t))\, f_j(\vec{x})}_{\prod_i \delta(y_i - x_i(t))\, f_j(\vec{y})} + \prod_i \delta(y_i - x_i(t))\,\eta_j(t) - \sigma \frac{\partial}{\partial y_j}\prod_i \delta(y_i - x_i(t))\right\} \qquad (1)$$

Note that
$$\left\langle \prod_i \delta(y_i - x_i(t))\,\eta_j(t)\right\rangle \overset{\text{Itō}}{=} \underbrace{\left\langle \prod_i \delta(y_i - x_i(t))\right\rangle}_{P(\vec{y})} \underbrace{\langle \eta_j(t)\rangle}_{0}$$

From (1), taking an average over the noise realisations

$$\frac{d}{dt}\left\langle \prod_i \delta(y_i - x_i(t))\right\rangle = -\sum_j \frac{\partial}{\partial y_j}\left\{ \left\langle \prod_i \delta(y_i - x_i(t))\right\rangle f_j(\vec{y}) + 0 - \sigma\frac{\partial}{\partial y_j}\left\langle \prod_i \delta(y_i - x_i)\right\rangle\right\}$$

$$\frac{d}{dt} P(\vec{y}) = -\sum_j \frac{\partial}{\partial y_j}\left[f_j(\vec{y})\, P(\vec{y}) - \sigma \frac{\partial}{\partial y_j} P(\vec{y})\right]$$

$$= -\vec{\nabla}\cdot\left[\vec{f}(\vec{y})\,P(\vec{y}) - \sigma\,\vec{\nabla}P(y)\right]$$

Comment: if $\sigma_{ij} = \sigma\,\delta_{ij}$, then Itô formula can be written as

$$\frac{d}{dt}\,g(\vec{x}(t)) = \sum_i \frac{\partial}{\partial x_i}\left(g(x)\right)\dot{x}_i + \frac{\partial^2}{\partial x_i^2}\left[\sigma\,g(\vec{x})\right]$$

$$= \vec{\nabla}g\cdot\dot{\vec{x}} \qquad + \Delta\left[\sigma g\right]$$

$$\frac{d}{dt}P(\vec{y}) = \frac{d}{dt}\left\langle \delta(\vec{y}-\vec{x}_t)\right\rangle_{\vec{x}_t} = \left\langle \underbrace{\vec{\nabla}_x\cdot\left[\delta(\vec{y}-\vec{x}_t)\right]}_{-\vec{\nabla}_y\,\delta(\vec{y}-\vec{x}_t)}\dot{\vec{x}}_t + \sigma\,\Delta\,\delta(\vec{y}-\vec{x}_t)\right\rangle$$

$$= \left\langle -\vec{\nabla}_y\,\delta(\vec{y}-\vec{x}_t)\cdot\vec{f}(\vec{x}_t) + \sigma\,\Delta\,\delta(\vec{y}-\vec{x}_t)\right\rangle$$

$$= -\vec{\nabla}_y\cdot\underbrace{\underbrace{\left\langle \delta(\vec{y}-\vec{x}_t)\,f(\vec{x}_t)\right\rangle}_{\left\langle \delta(\vec{y}-\vec{x}_t)\,f(\vec{y})\right\rangle}}_{\underbrace{\left\langle \delta(\vec{y}-\vec{x}_t)\right\rangle}_{P(\vec{y})}\cdot f(\vec{y})} + \sigma\,\Delta\left\langle \delta(\vec{y}-\vec{x}_t)\right\rangle$$

$$\frac{d}{dt}P(\vec{y}) = -\vec{\nabla}_y\cdot\left[f(\vec{y})\,P(\vec{y})\right] + \sigma\,\Delta\,P(\vec{y})$$

c) Application to ABPs

$$\dot{\vec{r}} = v_0\,\vec{u}(\theta) - \mu\vec{\nabla}V(\vec{r}) + \sqrt{2D_t}\,\vec{\eta} \quad ; \quad \dot{\theta} = \sqrt{2D_r}\,\xi$$

$$\dot{x} = v_0\cos\theta - \mu\,\partial_x V + \sqrt{2D_t}\,\eta_x$$

$$\dot{y} = v_0\sin\theta - \mu\,\partial_y V + \sqrt{2D_t}\,\eta_y$$

where ξ, η_x, η_y are 3 independent centered GWN of unit variance.

Fokker-Planck formula:

$$\frac{d}{dt}P(x,y,\theta,t) = -\frac{\partial}{\partial x}\left[(v_0\cos\theta - \mu\,\partial_x V)P\right] + D_t\,\frac{\partial^2}{\partial x^2}P$$

$$- \frac{\partial}{\partial y}\left[(v_0 \sin\theta - \mu \partial_y V) P \right] + D_t \frac{\partial^2}{\partial y^2} P$$

$$+ D_r \frac{\partial^2}{\partial \theta^2} P$$

$$= - \vec{\nabla}_{\vec{r}} \cdot \left[(v_0 \vec{u}(\theta) - \mu \vec{\nabla}_{\vec{r}} V) P \right] + D_t \Delta_{\vec{r}} P + D_r \partial_{\theta\theta} P$$

II Jump process & Master equations

II.1) discrete - state space

N configurations $\{\varphi\}$; Rate $W(\varphi \to \varphi')$

proba to go from φ to φ' in $[t, t+dt]$ $\underset{dt \to 0}{\sim}$ $W(\varphi \to \varphi') dt$ don't care

Master equation $\frac{d}{dt} P(\varphi, t) = \dots \Rightarrow P(\varphi, t+dt) - P(\varphi, t) = dt[\dots] + o(dt)$

in $[t, t+dt]$ $\quad P(\varphi \to \varphi' \to \varphi'') = W(\varphi \to \varphi') dt \, W(\varphi' \to \varphi'') dt \sim dt^2$

$\Rightarrow$ never have to go consider TWO processes.

$P(\varphi, t+dt) =$ [Proba to be in φ at t and to stay there] (1)

+ [Proba to be in any $\varphi' \neq \varphi$ and to go from φ' to φ] (2)

$$(2) = \sum_{\varphi' \neq \varphi} P(\varphi', t) W(\varphi' \to \varphi) dt$$

$$(1) = P(\varphi, t) \times (1 - \text{proba to go from } \varphi \text{ to any } \varphi' \neq \varphi) = P(\varphi, t)\left[1 - \sum_{\varphi'} W(\varphi \to \varphi') dt\right]$$

$$P(\varphi, t+dt) - P(\varphi, t) = dt \sum_{\varphi' \neq \varphi} W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi)$$

$$\frac{d}{dt} P(\varphi, t) = \sum_{\varphi' \neq \varphi} W(\varphi' \to \varphi) P(\varphi') - W(\varphi \to \varphi') P(\varphi)$$

II.2) Tumble dynamics, the limit of continuous space

$\theta \in [0, 2\pi]$

Path 1: proba to be in $[\theta, \theta + d\theta] = p(\theta) d\theta$ density of proba

$p(\theta, t+dt) d\theta = \dots$

Path 2: "Be wise, discretize"

$$\theta_k = \frac{2\pi}{N}\cdot k \Rightarrow N \text{ values} \Rightarrow \frac{d}{dt} P(\theta_k) = \sum_{m\neq k} P(\theta_m)\, W(\theta_m \to \theta_k) - P(\theta_k) W(\theta_k \to \theta_m)$$

uniform, isotropic tumbles $W(\theta_m \to \theta_k) = \alpha \cdot \frac{1}{N}$

$$\frac{d}{dt} P(\theta_k) = \underbrace{\sum_m \frac{1}{N}}\left[\alpha P(\theta_m) - \alpha P(\theta_k)\right] \qquad P(\theta_k) \to P(\theta)\,\underline{\underline{d\theta}}$$

$$\frac{1}{2\pi}\sum_m \underbrace{\frac{2\pi}{N}}_{d\theta} \xrightarrow[N\to\infty]{} \frac{1}{2\pi}\int d\theta'$$

$N\to\infty$

$$\frac{d}{dt} P(\theta) = \frac{\alpha}{2\pi}\int d\theta' \left[P(\theta') - P(\theta)\right] = -\alpha P(\theta) + \frac{\alpha}{2\pi}\int d\theta'\, P(\theta')$$

III A mixture of both world

. diffusion AND tumble

① Langevin dynamics $\Rightarrow$ $\partial_t P = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} D_1 - f(x)\right] P \equiv -H_{FP}\cdot P \quad (*)$

$$H_{FP} = -\frac{\partial^2}{\partial x^2} D_1 + \frac{\partial}{\partial x} f(x)$$

$(*) \Leftrightarrow P(x, t+dt) = P(x,t) - dt\, H_{FP}^{x} P + o(dt^\alpha) \quad \alpha > 1$

$\dot{x} = $ [Langevin eq°] ; $\theta \xrightarrow{\alpha} \theta'$

$$P(x,\theta_k, t+dt) = \int dx'\, P(x',\theta_k,t)\; P[(x',\theta_k,t) \to (x,\theta_k, t+dt)] \qquad (1)$$

$$+ \sum_{m\neq k} \int dx'\, P(x',\theta_m,t)\; P[(x',\theta_m,t) \to (x,\theta_k,t+dt)] \qquad (2)$$

in dt, $p(\text{two tumbles}) \propto dt^2 \Rightarrow$ disregard $\Rightarrow$ in (1) $\Rightarrow$ no tumble

$P[(x',\theta_k,t) \to (x,\theta_k,t+dt)] = $ (proba no tumbles) $\times$ (proba that Langevin takes the system from $x' \to x$)

$$= (1-\alpha dt)\, \underbrace{P_{FP}(x,\theta_k, t+dt \mid x',\theta_k,t)}_{P_{FP}(x,\theta_k,t\mid x',\theta_k,t) + dt \frac{\partial}{\partial t} P_{FP}(x,\theta_k,t \mid x',\theta_k,t)}$$

$$\underbrace{\quad}_{\delta(x-x')} \qquad -H_{FP}^{x,\theta_n} \underbrace{P(x,\theta_n,t|x',\theta_n,t)}_{\delta(x-x')}$$

$$= (1-\alpha dt)\left[\delta(x-x') - dt\, H_{FP}^{x,\theta_n}\,\delta(x-x')\right] + o(dt)$$

$$= \left(1-\alpha dt - dt\, H_{FP}^{x,\theta_n}\right)\delta(x-x') + o(dt)$$

$$(1)\quad \int dx'\, P(x',\theta_n,t)\left[\left(1-\alpha dt - dt\, H_{FP}^{x,\theta_n}\right)\delta(x-x')\right] + o(dt)$$

$$= \left(1-\alpha dt - dt\, H_{FP}^{x,\theta_n}\right)P(x,\theta_n,t)$$

$$(2)\quad x',\theta_m \to \underbrace{y,\theta_m \to y,\theta_n}_{\text{proba} \sim \alpha dt} \to x,\theta_n$$

$$\int \left(1-\alpha dt - dt\, H_{FP}^{y,\theta_m}\right)\delta(y-x')\,\frac{\alpha}{N}dt\,\left(1-\alpha dt - dt\, H_{FP}^{x,\theta_n}\right)\delta(y-x)\,dy$$

$$\sim 1 \times \frac{\alpha}{N}dt \times 1\ \delta(x'-x)$$

$$(2):\ \sum_{m\neq n}\int dx'\, P(x',\theta_m)\,\frac{\alpha}{N}dt\,\delta(x-x') \sim \frac{dt}{2\pi}\alpha\int d\theta'\, P(x,\theta')$$

All in all:

$$P(x,\theta,t+dt) = (1-\alpha dt - dt H_{FP}^{x,\theta})\, \underbrace{P(x,\theta,t)} + \frac{\alpha\, dt}{2\pi}\int d\theta'\, P(x,\theta')$$

$$\frac{d}{dt}P(x,\theta,t) = -H_{FP}^{x,\theta}\, P(x,\theta,t) - \alpha P(x,\theta,t) + \frac{\alpha}{2\pi}\int d\theta'\, P(x,\theta')$$

with the proper dynamics for x, y

$$\frac{d}{dt}P(x,\theta,t) = -\frac{\partial}{\partial x}\left[v_0 \cos\theta\, P\right] - \frac{\partial}{\partial y}\left[v_0 \sin\theta\, P\right] - \alpha P + \frac{\alpha}{2\pi}\int d\theta'\, P(x,y,\theta')$$